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The response of non-relativistic confined systems

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Abstract.

We study the non-relativistic response of a 'diquark' bound by confining forces, for which perturbation theory in the interaction fails. As non-perturbative alternatives we consider the Gersch-Rodriguez-Smith theory and a summation method. We show that, contrary to the case of singular repulsive forces, the GRS theory can generally be applied to confined systems. When expressed in the GRS-West kinematic variable y, the response has a standard asymptotic limit and calculable dominant corrections of orders $1/q, 1/q^2$. That theory therefore clearly demonstrates how constituents, confined before and after the absorption of the transferred momentum and energy, behave as asymptotically free particles. We compare the GRS results with those for a summation method for harmonic and square well confinement and also discuss the convergence of the GRS series for the response in powers of $1/q$.

1. Introduction.

Consider inclusive scattering on a target of mass M_T with confined constituents. We focus on the response or structure function $S(q,\omega)$ and in particular on its limit, when for fixed Bjorken scaling variable $x=(q^2-\omega^2)/2M_T\omega$, the momentum-energy transfer $q,\omega\to\infty$. The description of the response in that limit requires a relativistic theory and one finds for instance in the parton model that the above limit is that of free constituents.

There remains the intriguing question how exactly a system, composed of confined constituents which can only distribute the transferred energy to internal excitations and target recoil but not to dissociation, responds asymptotically as if the constituents were free. It is then tempting to exploit the simplicity of non-relativistic (NR) dynamics, in the hope that it may illuminate some features of this, intrinsically relativistic problem.

A second incentive to use NR dynamics comes from the relative ease to describe, for systems bound by regular forces, the approach of the response to its asymptotic limit. The situation is different if those forces, either repulsive or attractive are singular. In particular perturbation theory in the interaction fails and non-perturbative approaches have to be invoked. We already know that for systems governed by forces which contain a strong short-range repulsion the asymptotic limit of the response exists, but differs from the same for quasi-free constituents[1]. In contradistinction, surprisingly little has been done in the case of singular attractive, i.e. confining forces, and those are the main topic of this note.

An example of such a NR approach is a recent study by Greenberg of the response of a 'di-quark' bound by a harmonic oscillator potential[2]. He found that, in accordance with the naive parton model, the asymptotic response $qS(q,x)$ in the limit $q\to\infty$, at fixed NR Bjorken scaling variable $x=q^2/2M\omega$ vanishes unless x, which is the 'quark' momentum fraction in the infinite momentum frame, equals the 'quark'-target mass ratio m_i/M. We shall revisit

Greenberg's example in the following.

We start in Section 2 by scanning NR descriptions on their ability to handle singular forces. Those theories routinely employ, instead of the energy transfer ω, a second kinematic variable y which differs from the above NR Bjorken variable. Then using the theory of Gersch, Rodriguez and Smith (GRS)[3] we illustrate and emphasize essential differences in the treatment of singular repulsive, and attractive forces producing confinement. We show that the GRS theory can handle the latter category and we compute the response of 'di-quarks' confined by an harmonic oscillator and by an infinitely deep well. In Section 3 we generalize a non-perturbative summation technique used by Greenberg [2], compute with it the same examples and compare the results. In addition we calculate the response for general forces in a quasi-classical method and show that the outcome of the summation method is just the GRS theory to order q^{-2}. Convergence conditions for the GRS series are discussed in Section 4. In Section 5 we compare the response, once expressed in terms of the GRS-West variable [4] and then using the NR Bjorken scaling variable, and discuss the difference in content.

2. The GRS series for singular forces.

We limit ourselves in the following to 'di-quark' targets with constituents of equal mass m. In the target rest system its response per particle, including recoil is

$$S(q,\omega) = \frac{1}{2}\sum_n |\mathcal{F}_{0n}(q)|^2 \delta(\omega - q^2/4m - E_{n0}), \tag{1}$$

where $\mathcal{F}_{0n}(q) =< 0|e^{i\vec{q}\vec{r}/2} + e^{-i\vec{q}\vec{r}/2}|n >$ and E_{n0} are, respectively, inelastic form factors and excitation energies. A formal summation over n in (1) leads to

$$S(q,\omega) = \frac{1}{2\pi}\mathrm{Im}\sum_{i,j=1,2} < \Phi_0|\exp(-i\boldsymbol{q}\boldsymbol{r}_i)G(\omega)\exp(i\boldsymbol{q}\boldsymbol{r}_j)|\Phi_0 >, \tag{2}$$

with particle and relative coordinates related by $\boldsymbol{r}_{1,2} = \pm\boldsymbol{r}/2$. The response above contains

$$G(\omega) = (\omega + E_0 - K - V - i\eta)^{-1}, \tag{3}$$

the Greens function of the system in terms of the kinetic energy of the 'quarks' K, the binding energy E_0 and the confining interaction V. Regarding the sums in (2) we recall that the coherent contributions with $i \neq j$ decrease with increasing q much faster than do incoherent terms with $i = j$ and the latter will henceforth be disregarded [4].

For non-singular regular forces one frequently expands the full Greens function (3) in a Born series in V, thereby using the free Greens function $G_0 = (\omega + E_0 - K)^{-1}$. The first term of that expansion is obtained by replacing $G \to G_0$ in Eq. (3) and describes a 'quark', bound before and free after the transfer of (q, ω).

An alternative approach is due to Gersch, Rodriguez and Smith (GRS), who showed that the (incoherent part of the) reduced response $\phi(q, y) \equiv (q/m)S(q, \omega)$ for a two-particle target interacting through local forces can be written as [3]

$$\phi(q,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dse^{-iys}\int d^3\boldsymbol{r}\Phi_0(\boldsymbol{r}-s\hat{\boldsymbol{q}})T_\sigma \exp\left[i\frac{m}{q}\int_0^s [K+V(\boldsymbol{r}-\sigma\hat{\boldsymbol{q}})-E_0]d\sigma\right]\Phi_0(\boldsymbol{r})d\boldsymbol{r} \tag{4}$$

Here

$$y = -\frac{q}{2} + \frac{m\omega}{q} \tag{5}$$

is the non-relativistic GRS-West variable [3,4], while T_σ in Eq. (4) is an operator prescribing σ-ordering. Expanding the exponential in Eq. (4) one obtains

$$\phi(q,y) = F_0(y) + (m/q)F_1(y) + (m/q)^2F_2(y) + \cdots \tag{6}$$

The first term of the GRS expansion is the asymptotic limit of the reduced response in terms of the single particle momentum distribution $n(p)$

$$F_0(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dse^{-iys}\int d^3\boldsymbol{r}\Phi_0(\boldsymbol{r}-s\hat{\boldsymbol{q}})\Phi_0(\boldsymbol{r}) = 2\pi\int_{|y|}^{\infty} n(p)pdp \tag{7a}$$

For use below we recall that in the derivation of (7a) one passes the step

$$F_0(y) = \frac{q}{m}\int n(p)\delta\left(\omega + \frac{\boldsymbol{p}^2}{2m} - \frac{(\boldsymbol{p}+\boldsymbol{q})^2}{2m}\right)d^3\boldsymbol{p} = 2\pi\int_{|y|}^{\infty} n(p)pdp, \tag{7b}$$

where the above δ-function describes energy conservation of a 'quark' which before and after the absorption of $(\boldsymbol{q}, \omega)$ has the energy of a free, on-shell particle.

For both attractive and repulsive singular interactions, a Born perturbation theory in V fails. We thus turn to the GRS series (6), first for singular *repulsive* forces. As an example we choose an overall, weak binding potential $V(\boldsymbol{r})$ with a strong, short range repulsion, which for fixed $\boldsymbol{b} = \boldsymbol{r}_{\perp}, (\hat{z} = \hat{q})$ is shown in Fig. 1 as function of z. For arguments of the wave functions $z - s$ and z on different sides of the hard core, the σ-integrand in (4) intersects the hard core region and the corresponding integral diverges. Consequently for singular repulsion there is no meaning to the GRS expansion (6). This does not rule out other non-perturbative approaches, notably those where a finite V_{eff} replaces the singular V [5]. One can in fact show that an asymptotic limit for the response $F_0(y)$ exists, but is not given by Eq. (7)[1].

Also for singular attractive, i.e. *confining* interactions (Fig. 2) the Born series does not exist and we now investigate whether for those the GRS expression is applicable. Consider first the wave function arguments $z - s$ and z in (4) for which $z_1 < z - s, z < z_2$, i.e. which lie between the classical turning points z_1, z_2. Then, although the depth V_0 as well as the ground state energy E_0 tend to $-\infty$, the difference $V(\boldsymbol{r} - \sigma\hat{\boldsymbol{q}}) - E_0$ remains finite and so is the $\boldsymbol{r}$ integral in (4). One reaches the same conclusion if one of the two arguments above lies outside that region. There $V(\boldsymbol{r} - \sigma\hat{\boldsymbol{q}}) - E_0 \to -E_0$ is unbounded, but one of the wave functions $\Phi_0(\boldsymbol{r} - s\hat{\boldsymbol{q}})$ or $\Phi_0(\boldsymbol{r})$ tends to 0. Since the $\boldsymbol{r}$ integral is finite, the same is the case with all coefficients $F_n(y)$ in the GRS series (6). This holds in particular for the asymptotic limit $F_0(y)$ which, contrary to the case of singular repulsive forces [1], retains the form (7) in terms of the single constituent momentum distribution [6].

The outcome above is surprising since one expects singular attractive and repulsive forces to show similar exceptional behavior (see Section 5). We conclude:

i) In contradistinction to the case of repulsive forces, for certain classes of singular attrac-

tive potentials the GRS expansion (6) for the response exists and the coefficient functions are finite.

ii) The result for the asymptotic limit of the response $F_0(y)$, Eq. (7), is the one for free on-shell partons, as if the infinite potential and binding energy compensate one another.

iii) When for progressively decreasing q, increasing distances are probed the corrections F_n, $n \geq 1$ grow in importance. One may then expect that qualitatively different behavior sets in only if $q\lambda \approx 1$, i.e. if the relevant distances in the inclusive scattering become of the order of a typical length λ of V. We shall show in Section 4 that for those values of $q\lambda$ the GRS no more converges.

2a. The GRS expansion for selected examples

We now explicitly demonstrate the applicability of the GRS series for confining potentials on examples of one-dimensional, two-particle targets, thereby confirming the above heuristic reasoning. The general expressions for the first coefficients are [3], or can be transformed to

$$F_0(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds e^{-iys}\int_{-\infty}^{\infty} dx \Phi_0(x-s)\Phi_0(x) = n(y) \tag{8a}$$

$$F_1(y) = \frac{i}{2\pi}\int_{-\infty}^{\infty} ds e^{-iys}\int_{-\infty}^{\infty} dx \Phi_0(x-s)\Phi_0(x)\int_0^s d\sigma[V(x-\sigma)-V(x)] \tag{8b}$$

$$\begin{aligned} F_2(y) = \frac{i^2}{2\pi}\int_{-\infty}^{\infty} ds e^{-iys}\int_{-\infty}^{\infty} dx \Bigg\{ \Phi_0(x-s)\Phi_0(x)\frac{1}{2}\left[\int_0^s d\sigma[V(x-\sigma)-V(x)]\right]^2 \\ - [\Phi_0''(x-s)\Phi_0(x) - \Phi_0(x-s)\Phi_0''(x)]\int_0^s d\sigma \frac{s-\sigma}{m}[V(x-\sigma)-V(x)]\Bigg\}, \end{aligned} \tag{8c}$$

with $\Phi_0'' = d^2\Phi_0/dx^2$. We now apply the above to harmonic confinement of the relative motion (Fig. 3). Denoting by $\beta = (m\omega_0/2)^{1/2}$ the inverse length parameter, one finds the following finite expressions for the first three coefficient functions (the two lowest order terms had been worked out before [7])

$$
\begin{aligned}
F_0(y) &= \frac{1}{\sqrt{\pi\beta^2}} \exp(-y^2/\beta^2) \\
(m/q)F_1(y) &= -\left(\frac{y}{q}\right)\left(1-\frac{2y^2}{3\beta^2}\right)F_0(y) \qquad (9)\\
((m/q)^2F_2(y) &= -\frac{1}{6}\left(\frac{\beta}{q}\right)^2\left(1-9\frac{y^2}{\beta^2}+8\frac{y^4}{\beta^4}-\frac{4}{3}\frac{y^6}{\beta^6}\right)F_0(y)
\end{aligned}
$$

Next we consider the more intricate case of an infintely deep square well $V(x) = V_0\theta(a-|x|)$, with $V_0 \to -\infty$. Details are presented in Appendix A and we present here only the results. With $\gamma(ay) = (\pi/2)^2 - (ay)^2$

$$
F_0(y) = \frac{\pi a}{2}\left(\frac{\cos(ay)}{\gamma(ay)}\right)^2 \qquad (10a)
$$

$$
(m/q)F_1(y) = \frac{2}{qa}\left[ay - \gamma(ay)\mathrm{tg}(ay)\right]F_0(y) \qquad (10b)
$$

$$
(m/q)^2F_2(y) = \frac{\pi^2\gamma(ay)}{4(qa)^2}\left[1-\mathrm{tg}^2(ay)+\frac{4ay\,\mathrm{tg}(ay)-1}{\gamma(ay)}-\frac{4a^2y^2}{\gamma^2(ay)}\right]F_0(y) \qquad (10c)
$$

Note the periodic vanishing of $F_0(y), F_1(y)$ for $ay_n = n\pi/2, n \geq 2$, and for the same y_n the unboundedness of the ratio $F_1(y)/F_0(y)$ (but not of $F_1(y)$ itself!). Those are characteristics of the special properties of the square well potential.

3. The non-perturbative summation method

3a. Development

The non-perturbative expression (1) for the (reduced) response is in general quite impractical, since for regular interactions spectra are predominantly in the continuum. It is indeed the simplicity of spectra and wave functions of some confining forces which enables use of, what shall be referred to as the summation method. The basic assumption to be made in (1) after use of (5) is

$$
\phi(q,y) = \frac{q}{2m}\sum_n |\mathcal{F}_{0n}(q)|^2\delta\left(\frac{yq}{m}+\frac{q^2}{4m}-E_{n0}\right) \to \frac{q}{2m}\left|\mathcal{F}_{o\nu}(q)\right|^2\mathcal{N}\left(n\to\nu(q,y)\right), \qquad (11)
$$

with $\mathcal{N}(n) = |dE_{n0}/dn|^{-1}$, the level density. It prescribes the replacement $n \to \nu(q,y)$ everywhere and subsequent replacement of the sum over discrete n in (1) an integral. We shall now apply (11) to the cases studied above.

3b. Results for selected examples

For the case of harmonic confinement

$$\nu(q,y) = (q^2/2\beta^2)(1+2y/q)$$
$$\mathcal{N}(n) = \omega_0 \tag{12}$$

which leads to

$$\phi(q,y) = \frac{1}{\sqrt{\pi\beta^2}}\left[1 - \frac{y}{q} + \frac{\beta^2}{2q^2}\left(\frac{3y^2}{\beta^2} - \frac{1}{3}\right) + \mathcal{O}(q^{-3})\right]\exp\left[-\frac{q^2}{2\beta^2}h(q,y)\right]$$
$$h(q,y) = -2y/q + (1+2y/q)\ln(1+2y/q) \tag{13}$$

For fixed y, Eq. (13) allows a large q expansion

$$\phi(q,y) = [\bar{F}_0(y) + (m/q)\bar{F}_1(y) + (m/q)^2\bar{F}_2(y) + \mathcal{O}(q^{-3})], \tag{14}$$

with $\bar{F}_i(y)$ coinciding with the GRS coefficients in Eq. (9).

A number of remarks are in order. First, after adjusting constants due to different definitions of ϕ, Eqs. (13) and (14) do not agree with the corresponding result which can be derived from Eq. (14), Ref. (2). The difference is due to disregard there of all but the first two factors in Stirling's formula $n! = e^{-n}n^n(2\pi n)^{1/2}(1 - 1/12n + \mathcal{O}(n^{-2}))$. Since from (12) $\sqrt{n} = (q/\beta)\sqrt{1/2\,[1+2y/q]}$, even in the asymptotic limit, $\bar{F}_0(y)$ should contain the correction $(2\pi n)^{1/2}$. It incidentally renders $\bar{F}_0 \propto \beta^{-1}$, the natural length scale for the harmonic oscillator, and not $\bar{F}_0 \propto \omega_0^{-1}$ as in Ref. (2). The same also affects the dominant coeffitients $\bar{F}_1, \bar{F}_2$: once the corrections are applied, the lowest three coefficients agree with GRS.

It is of course gratifying to see the correspondence of those results to $\mathcal{O}(q^{-3})$ by two methods, as different as the explicit summation in Eq. (11) and the expression (4). In fact, the agreement should not be taken lightly. On the one hand it brings to the fore the question of convergence of the GRS series (see Section 4) and on the other hand the replacement in Eq. (9) of a discrete sum over delta-functions by an integral. For regular forces with an overwhelmingly continuous spectrum, the above replacement seems justified, but this is not obvious for confining potentials with purely discrete spectra.

Next we turn to the case of a 'di-quark' confined by an infinitly deep, one-dimensional square well. Eq. (11) yields for that case

$$\phi(q,y) = \frac{a^2 q}{2\pi^2 \nu(q,y)} \sum_{n\geq 1} \left[|\mathcal{F}_{0n}^{(+)}(q)|^2 \delta\left(n - \frac{1}{2} - \nu(q,y)\right) + |\mathcal{F}_{0n}^{(-)}(q)|^2 \delta\left(n - \nu(q,y)\right) \right], \qquad (15a)$$

where $\gamma(z) = (\pi^2/4) - z^2$ and

$$\pi\nu(q,y) = (aq/2)\left(1 + 4y/q + (\pi/aq)^2\right)^{1/2} = (aq/2)\left(1 + 2y/q + 2\gamma(ay)/a^2q^2 + \mathcal{O}(q^{-3})\right) \qquad (15b)$$

where $\mathcal{F}_{0n}^{(\pm)}(q)$ are the inelastic density form factors, linking the ground state to the excited even and odd parity states $a^{-1/2}\cos\left[x\pi(n-\frac{1}{2})/a\right]$, respectively $a^{-1/2}\sin\left[x\pi n/a\right]$, for $n \geq 1$. Proceeding as in Section (3a) one finds after some algebra

$$\left|\mathcal{F}_{0n}^{(\pm)}(q)\right|^2 = \frac{\pi^2}{4} \left| \frac{\cos(aq/2 - \pi\nu(q,y))}{\left[aq/2 - \pi\nu(q,y)\right]^2 - \pi^2/4} \mp \frac{\cos(aq/2 + \pi\nu(q,y))}{\left[aq/2 + \pi\nu(q,y)\right]^2 - \pi^2/4} \right|^2 \qquad (16)$$

Substituting (16) into (15a) and using (15b), one obtains

$$\phi(q,y) = \frac{\pi a}{2}\left[1 - 2y/q + \mathcal{O}(q^{-2})\right]\left[D_1^2(q,y) + D_2^2(q,y)\right] \qquad (17a)$$

$$D_1(q,y) = \frac{\cos(ay)}{\gamma(ay)}\left[1 + \frac{2y}{q}\left(1 - \gamma(ay)\frac{\mathrm{tg}(ay)}{2ay}\right) + \mathcal{O}(q^{-2})\right] \qquad (17b)$$

$$D_2(q,y) = \frac{\cos[a(q + y + \mathcal{O}(q^{-1}))]}{\pi^2/4 - a^2(q + y + \mathcal{O}(q^{-1}))^2} \qquad (17c)$$

Clearly for fixed y, $D_1(q,y)$ permits a large q expansion but $D_2(q,y) \propto \cos(aq)/q^2$ does not, thus

$$\phi(q,y) = [\bar{F}_0(y) + (m/q)\bar{F}_1(y) + \mathcal{O}(q^{-2})] + \frac{a\pi}{2} D_2^2(q,y) \qquad (18)$$

One then shows that $\bar{F}_{0,1}(y) = F_{0,1}(y)$ as in Eqs. (10), but for $n \geq 2$, $\bar{F}_n(y) \neq F_n(y)$. This is not surprising since the Euler interpolation formula, the first term of which gives the replacement of the sum in (11) by an integral, is only valid for analytic functions. Moreover, for a square well the density of levels $\mathcal{N}(n)$ grows linearly with n, casting doubt on the appropriate use of the summation method.

3c. The quasi-classical response for general V.

We prove in Appendix B the following quasi-classical result for the reduced response

$$\phi(q,y) = [F_0(y) + (m/q)F_1(y) + \mathcal{O}(q^{-2})] \tag{19}$$

It shows that the application of (11) generallyleads to the first two terms in the GRS series in the form Eqs. (8a), (8b). Clearly the above holds only if the quasi-classical method is at all applicable. This is for instance not the case for the square well treated in the previous section. When nevertheless worked out for that case, a non-analytic term like D_2 in (17c) appears also in this treatment.

The above result brings to mind Rosenfelder's treatment of the response using Wigner distribution functions [8]. It had been observed before that the approximation which Rosenfelder suggested and which uses another aspect of the semi-classical approach [9], also produces F_0 and the correct dominant correction $F_1(y)$, but not higher order coefficients.

4. Convergence of series expansions fo the response

Little is known on the convergence of various series expansions for the response. We mention a proof that the reduced response, when expressed in an alternative 'plane wave' kinematic variable y_0 instead of y, Eq. (5), converges to the plane wave impulse limit $\phi^{PWIA}(q,y_0)$ (and in fact to the asymptotic limit $F_0(y_0)$, Eq. (7)) provided the interaction has finite norm $||V||$ [10]. This sufficient condition does not distinguish between attractive and repulsive forces and excludes singular V of either type.

In the light of the above stands the remarkable observation above that for classes of confining forces, the exponent in the GRS expression (4) for the response exists. Again this is a necessary but not a sufficient condition for the convergence of the $1/q$ expansion (6). No doubt that for each system there are additional conditions which depend on dimensionless quantities. Those can be constructed from the external momenta y, q and lengths λ in the interaction V.

In fact the two examples treated are illuminating. First, for both one observes that

$\phi_n(q,y) \equiv (m/q)^n F_n(y)$ is independent of m. It had been remarked before, that although naturally appearing in the GRS theory [3], the ratio m/q cannot be an expansion parameter[11]: As the above results (9) and (10) show, the explicit mass of the constituents appears to cancel out in $\phi(q,y)$, but it may well be implicit in length parameters like $\lambda = (m\omega_0)^{-1/2}$ for the harmonic oscillator.

We now focus on F_1, F_2 in Eqs. (9) and (10) which dominate the approach to the asymptotic limit F_0 for non-vanishing, not too large y. We concentrate on $y = 0$ for which $F_1(0) = 0$ and the convergence is fastest. For the two examples considered, one has

$$(m/q)^2 F_2(0)/F_0(0) = \begin{cases} -(\beta^2/6q^2), & \text{for HO} \\ \pi^2(\pi^2-4)/16q^2a^2, & \text{for SqW} \end{cases} \tag{20}$$

The right hand side gives the size of corrections to the latter, governed by $q\lambda$. The condition $q\lambda >> 1$ coincides with the condition, already mentioned in the paragraph before Section 2b. For small, finite y one has to add $y/q << 1$.

5. Response of confined systems in terms of the Bjorken variable.

Until here we studied the reduced response expressed in terms of the NR GRS-West variable (5). In his treatment of NR harmonic confinement Greenberg used instead a NR Bjorken scaling variable

$$x = q^2/4m\omega \tag{21}$$

with

$$y = (q/4)(x^{-1} - 2) \tag{21'}$$

giving the relation to the GRS-West variable. One then shows that for harmonic confinement the summation method produces for large q

$$\tilde{\phi}(q,x) \equiv (q/m)S(q,\omega) = z(x,q)e^{-(q/q_0)(x^{-1}-2)^2}, \tag{22}$$

with z a regular function of $1/q$. The corresponding reduced response, when expressed as

function of x, has a vanishing asymptotic limit, unless $x = 1/2$, [2]. The latter is for the equal mass case the momentum fraction of the 'quarks' in the Galilei-boosted, infinite momentum frame. We now show that the same conditioned vanishing in fact holds for any interaction, regular or confining.

Using (21′), the asymptotic limit (7b) for $q \to \infty$ can as follows be expressed in x

$$qF_0[y(q,\omega)] = q\tilde{F}(q,x) = 4\int d\boldsymbol{p}\, n(p)\delta\left(x^{-1} - 2 - \frac{4p_z}{q}\right) \to \delta\left(x - \frac{1}{2}\right), \tag{23}$$

where $n(p)$ drops out due to its normalization. Therefore, starting from (7b) which is valid for the above forces, is the response for free, on-shell particles, one reaches the last identity, namely an asymptotic response with zero support, except for $x = 1/2$, [4]. We now ask whether the converse is also true. By way of example we take a constituent which before absorption of q is off-shell with energy $e(p) = p^2/2m + \mathcal{V}(p)$: $q\tilde{F}(q,x)$ has for $q \to \infty$ the *same* asymptotic limit $\delta(x - 1/2)$. One thus concludes that the response in the x variable of the form (23) is no evidence of asymptotically free, on-shell particles, whereas this is the case for the same in the y variable (7).

The poor content of the asymptotic limit of the response in terms of the NR Bjorken variable (21) contrasts with the same in y, Eq. (7), which as function of y enables the extraction of the momentum distribution of the constituents and the study of the approach towards that limit. In contradistinction, Eq. (22), expressed in the NR Bjorken variable, does not permit a series expansion in $1/q$. Therefore, no matter what V is, the use of y is preferable over the NR Bjorken variable x [4].

We close with a remark on the limited, singular support of the asymptotic limit of the reduced response in the NR Bjorken variable. Clearly any p-dependent term in the δ argument in (23) which does not vanish for asymptotic q, produces a finite support. For instance, using relativistic kinematics ($e_p = \sqrt{p^2 + m^2}$) in (7b) as well as the relativistic Bjorken variable $x_r = (q^2 - \omega^2)/(4m\omega)$ produces a proper x-support.[4]

6. Summary

We discussed above the non-relativistic reponse or structure function of two-body systems of confined constituents. Due to their singular nature, perturbation theory in V fails and non-perturbative methods are called for. We have investigated the GRS theory, which leads to a formally exact series expansion for the reduced response in powers of $1/q$. We could demonstrate that, contrary to the case of singular repulsive forces, for a system with singular confining forces that theory may make sense. As a consequence it permits, for fixed GRS-West variable y, a power series expansion in $1/q$. In particular the asymptotic limit is shown to be the one for free, on-shell constituents: For the smallest distances probed the response is just not sensitive to confinement of finite range. Higher order coefficients, relevant for increasing probed distances, correct the asymptotic limit as if the basic forces were regular.

A second method utilizes the occasional simplicity of spectra and wave functions of confined systems and calculates the response in the form Eq. (1) by an explicit summation over intermediate excited states. We then compared the outcome for the response in the two methods for examples of targets with confined constituents. In addition we showed that, whenever applicable, the semi-classical response agrees with the GRS series to $\mathcal{O}(q^{-2})$.

Next we tested the GRS series on its convergence in particular for $y = 0$. Convergence conditions require y to be small compared to typical inverse lenghts in V. Finally we compared the above responses if the GRS-West kinematic variable y is replaced by the non-relativistic Bjorken scaling variable x. The asymptotic limit vanishes except for values of x, equal to the momentum fraction of the constituents in the Galilean-boosted, infinite momentum frame. No additional information is contained in that limit, in contradistinction to the one in the GRS-West variable which contains the single constituent momentum distribution. For non-relativistic dynamics the above clearly favors the use of the GRS-West variable over the NR Bjorken variable.

Our concluding remark regards a conjecture of Greenberg, holding that a Gaussian decrease with q of $\phi(q,x)$ around $x = 1/2$ reflects the rapid vanishing of the 'quark-quark' interaction for decreasing separation [2]. It is instructive to transcribe the above behavior, using y instead of x. Thus $\tilde{\phi}(q,x) \to \phi(q,y) \propto \exp[-(y/\beta)^2]$, i.e. a Gaussian in y. We now claim that such behavior need not at all be related to inter-constituent forces vanishing with r: As an example we consider liquid ^{4}He with overall weak, attractive inter-atomic force with a very strong, short-range repulsion. The resulting single particle momentum distribution close to $T = 0°$ is roughly Gaussian [12] and so is the asymptotic response $F_0(y)$ as is also the case for harmonic confining forces (cf. Eq. (8a)).

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Appendix

A. Terms in the GRS series for an infinitly deep square well

The ground state wave function for a square well potential $V(x) = V_0\theta(a - |x|)$ is

$$\Phi_0(x) = \begin{cases} (1/\sqrt{a})\cos(k_0 x), & \text{for } |x| \le a \\ (k_0/\sqrt{ma|V_0|})\exp\left[-\sqrt{m|V_0|}(|x|-a)\right], & \text{for } |x| > a \end{cases} \tag{A1}$$

where $k_0 = \sqrt{m(E_0 - V_0)} \to \pi/2a$ for $V_0 \to -\infty$. Substituting the above Φ_0 in Eq. (8a) one finds in the limit $V_0 \to \infty$

$$F_0(y) = \frac{\pi a}{2}\left[\frac{\cos(ay)}{\gamma(ay)}\right]^2, \tag{A2}$$

where $\gamma(ay) = \pi^2/4 - a^2y^2$. All higher order coefficients contain the singular potential. Notice that in the limit $V_0 \to -\infty$, the integrands in Eqs. (8b), (8c) vanish for $|s| > 2a$, since the product of both wave functions decreases there exponentially with $|V_0|$. In general $F_n(y)$ draws only on that interval

$$F_n(y) = \frac{i^n}{2\pi}\int_{-2a}^{2a} ds e^{-iys} R_n(s), \tag{A3}$$

where $R_n(s)$ denote x-integrals (cf. Eqs. (8b), (8c)). Consider first the x-integration in Eq.

(8b) over the interval $0 \le s \le 2a$. For $-a+s \le x \le a$ the difference $V(x-\sigma) - V(x) = 0$ and the integral over the remaining x-sections can be written as

$$\begin{aligned} R_1(s) =& \left[\int_{-\infty}^{-a+s} dx + \int_a^\infty dx\right] \Phi_0(x-s)\Phi_0(x) \int_0^s d\sigma [V(x-\sigma) - V(x)] \\ =& V_0 \int_{-\infty}^{-a+s} \Phi_0(x-s)\Phi_0(x)(x+a-s)dx + V_0 \int_a^\infty \Phi_0(x-s)\Phi_0(x)(s+a-x)dx \end{aligned} \tag{A4}$$

Integrating by parts, one finds that only the second integral contributes in the limit $V_0 \to -\infty$

$$\lim_{V_0 \to -\infty} R_1(s) = -\frac{k_0 s}{ma} \sin(k_0 s) \tag{A5}$$

Likewise for the second interval $-2a \le s \le 0$

$$\lim_{V_0 \to -\infty} R_1(s) = \frac{k_0 s}{ma} \sin(k_0 s) \tag{A6}$$

Substitution of Eqs. (A5), (A6) into Eq. (A3) produces

$$(m/q)F_1(y) = \frac{1}{qa} \left[ay - \gamma(ay)\text{tg}(ay)\right] F_0(y) \tag{A7}$$

In a similar way one computes the third GRS coefficient function $F_2(y)$, Eq. (8c). For $0 \le s \le 2a$ only the interval $x > a$ contributes to $R_2(s)$ (cf. Eq. (A4))

$$\begin{aligned} R_2(s) =& V_0^2 \int_a^\infty \Phi_0(x-s)\Phi_0(x) \frac{(s+a-x)^2}{2} dx \\ & -V_0 \int_a^\infty [\Phi_0''(x-s)\Phi_0(x) - \Phi_0(x-s)\Phi_0''(x)] \frac{(s+a-x)^2}{2m} dx \end{aligned} \tag{A8}$$

Since for $x \ge a$ one has $\Phi_0''(x) = -mV_0\Phi_0(x)$, the first integral in Eq. (A8) proportional to V_0^2 cancels against the last term in the second integral. Consequently only the first term in the second integral survives. Integration by parts gives

$$\lim_{V_0 \to -\infty} R_2(s) = -\frac{{k_0}^3 s^2}{2m^2 a} \sin(k_0 s) \tag{A9}$$

As was the case for R_1 above (cf. Eqs. (A5) and (A6)), the region $-2a \le s \le 0$ produces (A9) but with the opposite sign. Substituting $R_2(s)$ into Eq. (A3) yields

$$(m/q)^2 F_2(y) = \frac{\pi^3 \gamma(ay)}{(4qa)^2} \left[1 - \text{tg}^2(ay) + \frac{4ay\,\text{tg}(ay) - 1}{\gamma(ay)} - \frac{4a^2y^2}{\gamma^2(ay)}\right] F_0(y) \tag{A10}$$

Eqs. (A2), (A7) and (A10) are the results cited in Eq. (10).

B. The quasi-classical response.

Neglecting wave functions of excited states in the classically forbidden region, we have inside the classical turning points x_1, x_2

$$\begin{aligned}\Phi_n(x) &\approx \sqrt{\frac{m}{\pi p_n(x)\mathcal{N}(n)}}\cos\left[\int_{x_1}^{x} d\xi p_n(\xi) - \frac{\pi}{4}\right] \\ p_n(x) &= \sqrt{m(E_n - V(x))} \to \frac{q}{2}\left[1 + \frac{2y}{q} + \frac{2(mE_0 - y^2)}{q^2} - \frac{2mV(x)}{q^2} + \mathcal{O}(q^{-3})\right],\end{aligned} \tag{B1}$$

with $\mathcal{N}$ as in Eq. (11) the density of states. In line with the summation method, we used above the δ function in (1) and the definition (5) of y. Aiming at a calculation to $\mathcal{O}(q^{-2})$, one finds for the reduced response (11)

$$\begin{aligned}\phi(q,y) = \frac{2}{\pi}\left(1 - \frac{2y}{q}\right)\Bigg|\int_{x_1}^{x_2} dx\Phi_0(x)e^{i\frac{qx}{2}}\frac{1}{2}\Bigg\{\exp\Bigg[-i\frac{q}{2}\left(1 + \frac{2y}{q} + \frac{2mE_0 - 2y^2}{q^2}\right)(x - x_1) \\ +i\frac{m}{q}\int_{x_1}^{x} d\xi V(\xi) + i\frac{\pi}{4}\Bigg] + \text{c.c.} + \mathcal{O}(q^{-2})\Bigg\}\Bigg|^2,\end{aligned} \tag{B2}$$

where the density of states cancels. Consider first contributions which come from the second ('c.c.') term in the above bracket. It is readily seen that those contribute to ϕ terms proportional to the elastic form factor or its square. The former decreases normally as $1/q^2$ and can be neglected in comparison with the first term in the brackets in (B2) [13]. To the desired order

$$\begin{aligned}\phi(q,y) = \frac{1}{2\pi}\int\int dx\,dx'\,\Phi_0(x)\Phi_0(x')\Bigg[1 - \frac{2y}{q} - i\frac{m}{q}(x - x')\left(E_0 - \frac{y^2}{m}\right) \\ +i\frac{m}{q}\int_{x'}^{x} d\xi V(\xi) + \mathcal{O}(q^{-2})\Bigg]e^{iy(x'-x)}\end{aligned} \tag{B3}$$

Using $-y^2/m = (1/m)(d^2/dx^2)e^{-iyx}$ and the Schroedinger equation, and integrating by parts one finds

$$\phi(q,y) = \frac{1}{2\pi}\int\int dx\,dx'\,\Phi_0(x)\Phi_0(x')\left[1 + i\frac{m}{q}\int_{x'}^{x} d\xi V(\xi) - i\frac{m}{q}(x - x')V(x) + \mathcal{O}(q^{-2})\right]e^{iy(x-x')} \tag{B4}$$

Writing $s = x - x'$ and replacing $\xi \to x - \sigma$ one shows that (B4) is Eq. (19) with $F_{0,1}(y)$ as in Eqs. (8).

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[13]The square well is a notable exception and leads to slower decreasing and oscillating contributions like D_2^2 in (18).

Figure captions.

Fig. 1. Cut of a weakly binding potential $V(\boldsymbol{b}, z)$ with strong short range repulsion as function of z for fixed b.

Fig. 2. Same as Fig. 1 for a confining $V(\boldsymbol{b}, z)$.

Fig. 3. Harmonic confining potential energy E_0, V is defined as the $V_0 \to -\infty$ limit of $V(x) = m\omega_0^2 x^2/4 + V_0$ for $|x| < 2\left(|V_0|/m\omega_0^2\right)^{1/2}$ and $V = 0$ for $|x| \geq 2\left(|V_0|/m\omega_0^2\right)^{1/2}$ and $E_0 = V_0 + \omega_0/2$. V_0 drops out of the expressions (8).